ON THE GROMOV WIDTH OF HOMOGENEOUS KÄHLER MANIFOLDS

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ABSTRACT. We compute the Gromov width of homogeneous Kähler manifolds with second Betti number equal to one. Our result is based on the recent preprint [4] and on the upper bound of the Gromov width for such manifolds obtained in [6].

1. INTRODUCTION

The Gromov width [3] of a $2n$-dimensional symplectic manifold (M, ω) is defined as

$$c_G(M, \omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega)\}, \tag{1}$$

where

$$B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} x_j^2 + y_j^2 < r^2\} \tag{2}$$

is the open ball of radius r endowed with the standard symplectic form $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ of $\mathbb{R}^{2n}$. By Darboux's theorem $c_G(M, \omega)$ is a positive number. Computations and estimates of the Gromov width for various examples have been obtained by several authors (see, e.g. [6] and references therein). The main result of this paper is the following theorem proved in the next section.

Theorem 1. *Let (M, ω) be a compact homogeneous Kähler manifold such that $b_2(M) = 1$ and ω is normalized so that $\omega(A) = \int_A \omega = \pi$ for the generator $A \in H_2(M, \mathbb{Z})$. Then*

$$c_G(M, \omega) = \pi. \tag{3}$$

The class of manifolds in Theorem 1 includes all Hermitian symmetric space of compact type whose Gromov width has been computed in [5]. We do not know if the assumption on the second Betti number can be dropped.

2. Proof of Theorem 1

The proof of Theorem 1 is mainly based on the lower bound recently obtained by K. Kaveh [4]:

Theorem A *Let X be a smooth complex projective variety embedded in a complex projective space $\mathbb{CP}^N$. Then*

$$c_G(X, \omega_{FS}) \geq 1, \tag{4}$$

where ω_{FS} denotes the restriction to X of the Fubini–Study Kähler form of $\mathbb{CP}^N$.

Proof of Theorem 1 The upper bound $c_G(M, \omega) \leq \pi$ is Theorem 1 in [6]. In order to obtain the lower bound $c_G(M, \omega) \geq \pi$, consider the integral Kähler form $\hat{\omega} = \frac{\omega}{\pi}$ on M. Let (L, h) be the holomorphic hermitian line bundle on M such that $\mathrm{Ric}(h) = \hat{\omega}$, where $\mathrm{Ric}(h)$ is the 2-form on M defined by $\mathrm{Ric}(h) = -\frac{i}{2\pi} \partial \bar{\partial} \log h(\sigma, \sigma)$, for a local trivializing holomorphic section σ of L. Let $s_0, \ldots, s_N$ be an orthonormal basis for the space of global holomorphic sections $H^0(L)$ of L equipped with the L^2-scalar product $\langle \cdot, \cdot \rangle$ given by:

$$\langle s, t \rangle = \int_M h(s, t) \frac{\hat{\omega}^n}{n!}, \quad s, t \in H^0(L).$$

Then, it is not hard to see (see, e.g. [1]), due to the homogeneity and simply connectedness of M, that the Kodaira map $k : M \to \mathbb{CP}^N, x \mapsto [s_0(x) : \cdots : s_N(x)]$ is a Kähler immersion, i.e. $k^* \omega_{FS} = \hat{\omega}$. Moreover, in [2, Theorem 3] is proved that such a map is injective, and hence

$(M, \hat{\omega})$ is symplectomorphic to $(k(M), \omega_{FS})$. By Theorem A and by the conformality of the Gromov width one gets

$$c_G(M, \omega) = \pi c_G(M, \hat{\omega}) = \pi c_G(k(M), \omega_{FS}) \geq \pi$$

and the theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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